もうひとつの $\sum_{k=1}^{n} k^2$ の求め方 (1)

証明は自分で考えてね

$$\sum_{k=1}^{n} 1 = n$$
$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$
$$\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2)$$
$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$
$$\sum_{k=1}^{n} k(k+1)(k+2)(k+3) = \frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$$

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} k(k+1) - \sum_{k=1}^{n} k \\
&= \frac{1}{3}n(n+1)(n+2) - \frac{1}{2}n(n+1) \\
&= \frac{1}{6}n(n+1)\{2(n+2) - 3\} \\
&= \frac{1}{6}n(n+1)(2n+1)
\end{aligned}$$

もうひとつの $\sum_{k=1}^{n} k^2$ の求め方 (2)

証明は自分で考えてね

$$\begin{array}{ccccccccccccc}
\mathbf{1} & & \mathbf{4} & & \mathbf{9} & & \mathbf{16} & & \mathbf{25} & & \mathbf{36} & & \cdots \\
& \mathbf{3} & & \mathbf{5} & & \mathbf{7} & & \mathbf{9} & & \mathbf{11} & & \cdots & \\
& & \mathbf{2} & & \mathbf{2} & & \mathbf{2} & & \mathbf{2} & & \cdots & & \\
& & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \cdots & & &
\end{array}$$

$$\begin{aligned}
&1+4+9+16+25+36+\cdots+n^2 \\
&= {}_n\mathrm{C}_1\cdot 1+{}_n\mathrm{C}_2\cdot 3+{}_n\mathrm{C}_3\cdot 2+{}_n\mathrm{C}_4\cdot 0+\cdots \\
&= n\cdot 1+\frac{n(n-1)}{2\cdot 1}\cdot 3+\frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}\cdot 2 \\
&= n+\frac{3n(n-1)}{2}+\frac{n(n-1)(n-2)}{3} \\
&= \frac{1}{6}n\left\{6+9(n-1)+2(n-1)(n-2)\right\} \\
&= \frac{1}{6}n(6+9n-9+2n^2-6n+4) \\
&= \frac{1}{6}n(2n^2+3n+1) \\
&= \frac{1}{6}n(n+1)(2n+1)
\end{aligned}$$

$$\begin{array}{ccccccccccccc}
\mathbf{1} & & \mathbf{8} & & \mathbf{27} & & \mathbf{64} & & \mathbf{125} & & \mathbf{216} & & \cdots \\
& \mathbf{7} & & \mathbf{19} & & \mathbf{37} & & \mathbf{61} & & \mathbf{91} & & \cdots & \\
& & \mathbf{12} & & \mathbf{18} & & \mathbf{24} & & \mathbf{30} & & \cdots & & \\
& & & \mathbf{6} & & \mathbf{6} & & \mathbf{6} & & \cdots & & & \\
& & & & \mathbf{0} & & \mathbf{0} & & \cdots & & & &
\end{array}$$

$$\begin{aligned}
&1+8+27+\cdots+n^3 \\
&= {}_n\mathrm{C}_1\cdot 1+{}_n\mathrm{C}_2\cdot 7+{}_n\mathrm{C}_3\cdot 12+{}_n\mathrm{C}_4\cdot 6+{}_n\mathrm{C}_5\cdot 0+\cdots \\
&= n\cdot 1+\frac{n(n-1)}{2\cdot 1}\cdot 7+\frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}\cdot 12+\frac{n(n-1)(n-2)(n-3)}{4\cdot 3\cdot 2\cdot 1}\cdot 6 \\
&= n+\frac{7n(n-1)}{2}+2n(n-1)(n-2)+\frac{n(n-1)(n-2)(n-3)}{4} \\
&= \frac{1}{4}n\left\{4+14(n-1)+8(n-1)(n-2)+(n-1)(n-2)(n-3)\right\} \\
&= \frac{1}{4}n(4+14n-14+8n^2-24n+16+n^3-6n^2+11n-6) \\
&= \frac{1}{4}n(n^3+2n^2+n) \\
&= \frac{1}{4}n^2(n+1)^2
\end{aligned}$$

$$\begin{array}{ccccccccccccccc}
0 & & S_1 & & S_2 & & S_3 & & S_4 & & S_5 & & S_6 & & \cdots \\
 & a_1 & & a_2 & & a_3 & & a_4 & & a_5 & & a_6 & & \cdots & \\
 & & b_1 & & b_2 & & b_3 & & b_4 & & b_5 & & \cdots & & \\
 & & & d & & d & & d & & d & & \cdots & & &
\end{array}$$

$$\begin{aligned}
S_1 &= a_1 \\
&= {}_1\mathrm{C}_1 a_1 \\
S_2 &= S_1 + a_2 \\
&= a_1 + a_2 \\
&= a_1 + a_1 + b_1 \\
&= 2a_1 + b_1 \\
&= {}_2\mathrm{C}_1 a_1 + {}_2\mathrm{C}_2 b_1 \\
S_3 &= S_2 + a_3 \\
&= 3a_1 + 3b_1 + d \\
&= {}_3\mathrm{C}_1 a_1 + {}_3\mathrm{C}_2 b_1 + {}_3\mathrm{C}_3 d
\end{aligned}$$

$$\begin{aligned}
S_4 &= S_3 + a_4 \\
&= 4a_1 + 6b_1 + 4d \\
&= {}_4\mathrm{C}_1 a_1 + {}_4\mathrm{C}_2 b_1 + {}_4\mathrm{C}_3 d \\
&\vdots \\
S_n &= S_{n-1} + a_n \\
&= {}_n\mathrm{C}_1 a_1 + {}_n\mathrm{C}_2 b_1 + {}_n\mathrm{C}_3 d
\end{aligned}$$

$$\boxed{a_n = {}_{n-1}\mathrm{C}_0 a_1 + {}_{n-1}\mathrm{C}_1 b_1 + {}_{n-1}\mathrm{C}_2 d \qquad n \geqq 3 \cdots ①}$$

$n=3$ のとき $a_3 = a_2 + b_2 = (a_1 + b_1) + (b_1 + d) = a_1 + 2b_1 + d$ で①は正しい。

$n=k$ のとき①は正しいと仮定する。

$a_k = {}_{k-1}\mathrm{C}_0 a_1 + {}_{k-1}\mathrm{C}_1 b_1 + {}_{k-1}\mathrm{C}_2 d \cdots ②$

$$\begin{aligned}
a_{k+1} &= a_k + b_k \\
&= {}_{k-1}\mathrm{C}_0 a_1 + {}_{k-1}\mathrm{C}_1 b_1 + {}_{k-1}\mathrm{C}_2 d + b_k \\
&= {}_{k-1}\mathrm{C}_0 a_1 + {}_{k-1}\mathrm{C}_1 b_1 + {}_{k-1}\mathrm{C}_2 d + a_1 + b_1 + (k-1)d \\
&= {}_{k-1}\mathrm{C}_0 a_1 + {}_{k-1}\mathrm{C}_1 b_1 + {}_{k-1}\mathrm{C}_2 d + a_1 + {}_{k-1}\mathrm{C}_0 b_1 + {}_{k-1}\mathrm{C}_1 d \\
&= {}_k\mathrm{C}_0 a_1 + {}_k\mathrm{C}_1 b_1 + {}_k\mathrm{C}_2 d
\end{aligned}$$

$n = k+1$ のとき正しい。

$$\boxed{S_n = {}_n\mathrm{C}_1 a_1 + {}_n\mathrm{C}_2 b_1 + {}_n\mathrm{C}_3 d \qquad n \geqq 3 \cdots ①}$$

$n=3$ のとき $S_3 = {}_3\mathrm{C}_1 a_1 + {}_3\mathrm{C}_2 b_1 + {}_3\mathrm{C}_3 d = 3a_1 + 3b_1 + d$ で①は正しい。

$n=k$ のとき①は正しいと仮定する。

$S_k = {}_k\mathrm{C}_1 a_1 + {}_k\mathrm{C}_2 b_1 + {}_k\mathrm{C}_3 d$

$$\begin{aligned}
S_{k+1} &= S_k + a_{k+1} \\
&= {}_k\mathrm{C}_1 a_1 + {}_k\mathrm{C}_2 b_1 + {}_k\mathrm{C}_3 d + {}_k\mathrm{C}_0 a_1 + {}_k\mathrm{C}_1 b_1 + {}_k\mathrm{C}_2 d \\
&= {}_{k+1}\mathrm{C}_0 a_1 + {}_{k+1}\mathrm{C}_1 b_1 + {}_{k+1}\mathrm{C}_2 d
\end{aligned}$$

$n = k+1$ のとき正しい。